

A NEW KINEMATICAL DEFINITION OF ORBITAL ECCENTRICITY

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SUMMARY: A new concept of orbital eccentricity is given. The dimensionless quantities proposed in the present paper to serve as orbital eccentricities have a kinematical nature. The purpose is to use them in describing the motion for the case of three-dimensional orbits. A comparison done for nearly planar orbits shows that the values of the eccentricities proposed here do not differ significantly from those corresponding to the eccentricities of geometric nature usually applied.

Key words. Galaxy: kinematics and dynamics

1. INTRODUCTION

The notion of orbital eccentricity has been present in astronomical literature since long ago. Usually this notion concerns the Keplerian motion and has a geometric meaning because the orbits of celestial bodies in the two-body problem or Keplerian motion are conic sections (e.g. Binney and Tremaine 1987, p. 107). In addition to this geometric meaning, there is a dynamical meaning since the orbital eccentricity serves as a suitable quantitative dimensionless measure of deviation from circular motion. The two-body problem is just a special case of steady state and spherical symmetry, which is also a special case of steady state and axial symmetry. Both cases are well known in stellar dynamics and admit the circular motion. For this reason, the notion of orbital eccentricity becomes important also in stellar dynamics, because deviations from the circular motion should be describable in a suitable way by means of a dimen-

sionless quantity, in the cases where the orbits are not simple conic sections. The orbits in the case of spherical symmetry are planar and the essential difference, in comparison to the Keplerian motion, is the general absence of the line of apsides. If the symmetry is axial, the orbits are three-dimensional. It is clear that, unlike the two-body problem where the geometric eccentricities of the conic sections are quite sufficient for the purpose of describing the deviations from the circular motion, in more general cases of stellar dynamics an orbital eccentricity defined geometrically does not always satisfy the requirements. Therefore, it is not surprising that several definitions of eccentricity have been offered as will be seen below. Bearing in mind, above all, the three-dimensional (3D) orbits characteristic for steady state and axial symmetry, the present author had another idea of introducing "new orbital eccentricities" which may be useful in analysing the motion of stars in systems of complex structure like galaxies.

2. CONCEPTS OF ORBITAL ECCENTRICITY

In the case of steady state and spherical symmetry the orbits are planar and the time dependence is regular (for details: e.g. Ninković and Jovanović 2008). Hence, the maximum and minimum distances to the system centre are well defined for a bound orbit. Therefore, it is not surprising that then the orbital eccentricity e_g is most frequently defined as

$$e_g = \frac{r_a - r_p}{r_a + r_p}, \quad (1)$$

where r_a and r_p are the maximum and minimum distances to the centre (apo- and pericentric distances), respectively. The advantage of this definition is that, in the special case of Keplerian motion, the eccentricity e_g coincides with the geometric eccentricity of an ellipse.

The eccentricity definition according to the formula (1) is, though used most frequently, not unique. There are also alternative definitions. For instance, one can introduce radius of the circular orbit having the same energy as the orbit under study A , and the radius of the circular orbit with the same angular momentum as the orbit under study p . Then the alternative orbital eccentricity e_c will be defined as (e.g. Kuzmin and Malasidze 1970, p. 195)

$$e_c = (1 - \frac{p}{A})^{1/2}. \quad (2)$$

As not difficult to verify, the eccentricity e_c defined in this way is equal to the eccentricity e_g (formula (1)) for the case of the Keplerian motion.

In the same paper Kuzmin and Malasidze (1970) give another possibility. This time the distance A , determined in the above way, is used so that one has the following formula

$$e_J = (1 - \frac{J^2}{A^2 u_c^2(A)})^{1/2}, \quad (3)$$

where J is the modulus of the specific (per unit mass) angular momentum and $u_c(A)$ the circular velocity at the distance A . If the motion is Keplerian, the two eccentricities, e_g and e_J , coincide.

Finally, one can mention the orbital eccentricity e_f introduced by Lynden-Bell (1963, formula (2) therein), also equal to e_g for the case of the Keplerian motion. The formula is

$$1 - e_f^2 = [1 + |\frac{W_r}{2\pi J}|]^{-2}. \quad (4)$$

Here W_r is the specific action, or conjugate momentum, as called by Lynden-Bell himself, in r , the instantaneous distance to the centre, and J is the modulus of the specific angular momentum as defined above. The specific action is obtained by integrating over one complete cycle of a given coordinate.

The equality to e_g in the case of the Keplerian motion clearly indicates that all these alternative orbital eccentricities are defined to be within the

limits 0 to 1, where the value of 0 means the circular motion, while the other limiting value, equal to 1, for the majority of spherical potentials different from that of a point mass usually means zero angular momentum, i.e. the test particle moves along a straight line through the centre of the system being still bound.

In the case of spherically symmetric potentials different from that of a point mass, for the same orbit the orbital eccentricities e_g , e_c , e_J and e_f will differ, but the differences are insignificant (Kutuzov 1985, 1987).

The situation is different with the axial symmetry. Firstly, the orbits are no longer planar. Furthermore, the question regarding the isolating integrals for the case of steady state and axial symmetry has not been clarified yet (e.g. Ninković and Jovanović 2009). The orbits for this case are usually described by using three elements, very often at least one of them being dimensionless. This element is in most cases the orbital eccentricity defined analogously as here in formula (1), but, bearing in mind that one deals with 3D orbits, the distances are those to the axis of galactic rotation (R_a and R_p) (e.g. Ossipkov and Kutuzov 1996, Altmann and de Boer 1999, Dinescu et al. 1999, Nordström et al. 2004, Vidojević and Ninković 2009). Authors computing 3D orbits often use another dimensionless element, a sort of ratio of the maximum distance to the galactic plane and the corresponding distance to the rotation axis R (e.g. Altmann and de Boer 1999, Dinescu et al. 1999, Vidojević and Ninković 2009). Vidojević and Ninković (2009) use two orbital eccentricities: the planar one which describes the deviation from the circular motion in the galactic plane and the vertical one describing such deviation perpendicularly to the galactic plane.

3. NEW KINDS OF ORBITAL ECCENTRICITY

The way of defining the planar eccentricity as used by Vidojević and Ninković (2009), and also by other authors before, is suitable in the case of nearly planar orbits, when the test particle remains all the time sufficiently close to the plane of symmetry (galactic plane in the case of Milky Way). These orbits admit a quasi independence of planar motions resulting in an adiabatic invariant concerning the motion in the z direction (e.g. Binney and Tremaine 1987, p. 181). For this reason their periodicities are rather well defined (e.g. Ninković and Jovanović 2009) and, consequently, notions and definitions used in the more special case of spherical symmetry can be successfully applied to such orbits. A general 3D orbit is chaotic; the time dependences of both R and z become very complicated (e.g. Ninković and Jovanović 2009). As a consequence the extremal values of these coordinates are no longer clearly defined. For instance, in the case of a nearly planar orbit, the extrema R_a and R_p repeat periodically, just like in the case of spherical symmetry, and $|z|_{\max}$ shows a weak dependence on R . In addition, the ratio $|z|_{\max}(R)/R$

is low then. Therefore, the two eccentricities, as defined in the preceding section, provide a suitable and satisfactory description of deviation from the circular motion. In the general case of a 3D orbit, the highest value of $|z|$ can exceed a corresponding value of R significantly (also the correspondence is dubious). One may conclude that 3D orbits which correspond to the case of steady state and axial symmetry require alternative definitions of orbital eccentricity, where orbital eccentricity is understood as a dimensionless quantity suitably describing the deviation from a circular orbit in the plane of symmetry for a 3D orbit.

In the paper by Ninković and Jovanović (2009), the difficulties concerning the number of independent isolating integrals for the case of steady state and axial symmetry were discussed. The authors propose the mean values over time (time interval very long, comparable to the age of a Milky-Way subsystem) of the velocity squares in R and z to be used as quasi integrals of motion. With regard to this, the present author proposes two orbital eccentricities of kinematical nature. For simplicity, they will be referred to as planar (e_R) and perpendicular (e_z). The former one is defined as

$$e_R^2 = \frac{\langle \dot{R}^2 \rangle_t}{\langle V^2 \rangle_t}, \quad (5)$$

where $\langle \dot{R}^2 \rangle_t$ is the mean of the radial velocity component square over time, whereas $\langle V^2 \rangle_t$ is the mean over time of the modulus of the galactocentric velocity square. These mean values are, of course, taken over the same time interval. Since in a general case of steady state and axial symmetry the periods in R and z are not clearly defined (e.g. Ninković and Jovanović 2009), this time interval has to be long enough, say comparable to the age of the Milky Way. More precisely, it should be equal to the Hubble time in the order of magnitude. Such a long time interval ensures that the means of velocity squares provide a correct description of an orbit. It is clear that the defined eccentricity cannot be smaller than 0 and greater than 1. Note that this is a cylindrical galactocentric reference frame, so that $V^2 = \dot{R}^2 + \Theta^2 + \dot{z}^2$.

Analogously, the perpendicular eccentricity is

$$e_z^2 = \frac{\langle \dot{z}^2 \rangle_t}{\langle V^2 \rangle_t}. \quad (6)$$

Thus, the formulae (5) and (6) do not yield the eccentricities themselves, but their squares. The reason is that it is desirable to have as close as possible agreement between the planar eccentricity (5) and its geometric analogue defined by (1), in which the quantities R_a and R_p are used. It is clear that such an agreement is meaningful only for nearly planar orbits. Taking advantage of the new definitions of orbital eccentricity, it is possible to introduce another eccentricity, kinematical in its nature, the spatial eccentricity which is obtained from these two

$$e_s^2 = e_R^2 + e_z^2. \quad (7)$$

It is clear that this spatial eccentricity is also limited to be between 0 and 1. The former possibility

means circular motion in the plane of symmetry, the latter one that the modulus of the specific-angular-momentum component J_z is equal to zero (due to axial symmetry J_z is constant). In this case the orbit would contain the z -axis. Similarly, $e_R = 1$ would mean the rectilinear motion through the centre in the plane of symmetry; $e_z = 1$ would also mean the rectilinear motion, but along the z axis.

4. DISCUSSION

The main purpose of introducing the eccentricities proposed in the preceding section is to suggest suitable dimensionless quantities describing the motion of a test particle on a 3D orbit for the case of steady state and axial symmetry. They are of kinematical nature. Therefore, no coincidence with the corresponding geometric eccentricities can be expected. A comparison between a geometric eccentricity and either of the two defined above ((5) and (6)) is meaningful only for nearly planar orbits. The corresponding planar geometric eccentricity, as already said above, would be defined in analogy with (1) via the extremal distances to the rotation axis, R_a and R_p . As for the perpendicular eccentricity, since the amplitude of the z motion, in a general case, depends on R , a ratio of the kind $|z|_{\max}(R)/R$ could be used. Implicitly, this would mean that the ratio $|z|_{\max}(R)/R$ is approximately constant; something that may be reasonable with regard to what is known about the structure of stellar discs (also Binney and Tremaine 1987, p. 181, formula (3-173)). The problem becomes more simple when epicyclic orbits are studied, because then, due to the very low planar eccentricity, the dependence $|z|_{\max}(R)$ practically vanishes and the ratio $|z|_{\max}/R_m$, where R_m is the arithmetic mean between R_a and R_p , appears as a reasonable choice.

A general comparison is impossible, as all the eccentricities, defined in any of the ways mentioned above, depend on the properties of the assumed potential. The exception is only an epicyclic orbit, because for a test particle always sufficiently close to a circle in the plane of symmetry, the time dependences are similar and do not depend much on the potential. Preliminary comparisons show that the planar and perpendicular eccentricities, as defined in Section 3, do not differ much from their geometric analogues for the case of nearly planar orbits.

The mean squares of velocity components over time, used to define the two orbital eccentricities in Section 3, come from quantities known as actions. In other words these mean squares are obtained by integrations. In this sense the eccentricities e_R and e_z are similar to Lynden-Bell's eccentricity e_f , because the latter is also obtained through actions which have the property of being adiabatically invariant (e.g. Binney and Tremaine 1987, p. 178). The case discussed by Lynden-Bell concerns the way how to calculate orbital elements for a planar orbit, but without using a complex numerical procedure which, at the time when Lynden-Bell proposed e_f (1963), required a

very long computing time. Now, when use of the numerical procedures in calculating 3D orbits is common, such a problem does not exist. In addition, it is also usual to use the energy integral for the purpose of controlling the accuracy and precision. This means that all quantities necessary for the calculation of e_R and e_z are obtained at each step. After this, nothing is easier than to calculate the quantities required in formulae (5) and (6).

It is clear that the eccentricity e_R can be used in the case of a purely planar motion (spherical symmetry) as well. In this case, the square of the velocity magnitude would contain only two components: the radial (\dot{r}^2) and the tangential (v_t^2) ones obtainable by dividing the modulus of the specific angular momentum by the instantaneous distance to the centre r . In principle, the ratio of the instantaneous \dot{r}^2 to the square of the corresponding spatial velocity v^2 can be used as a measure of the geometric orbital eccentricity e_g for an arbitrary spherically symmetric potential (Ninković 1986). Thus, a large fraction of the radial component in $\langle v^2 \rangle_t$ indicates a high geometric eccentricity e_g .

In the particular case of the point-mass potential - GM/r (G gravitational constant) - by using the formula yielding the mean value over time for v^2 (e.g. Ninković 1987) and the action in the position angle in the orbital plane ψ (Lynden-Bell 1963) one obtains the relation between e_g and e_R

$$e_R = [1 - (1 - e_g^2)^{1/2}]^{1/2}. \quad (8)$$

As easily seen, e_R and e_g coincide for two extremal values only, 0 and 1, whereas e_g is generally higher; at low e_g the relation $e_R \approx 0.71e_g$ holds, and as e_g increases, the ratio e_R/e_g becomes higher, to become almost 1 at $e_g \approx 0.99$.

Another particular case of interest may be that concerning the potential corresponding to the constant density within a sphere of finite radius (equal to $c_1 - c_2r^2$, c_1 and c_2 positive constants). Using again the same formulae (yielding the mean value over time for v^2 and the action in ψ) one obtains

$$e_R = \left[\frac{2e_g^2}{1 + e_g^2} \right]^{1/2}. \quad (9)$$

As easily seen, e_R and e_g coincide for two extremal values only, 0 and 1, whereas e_g is generally lower; at low e_g the relation $e_R \approx 1.41e_g$ holds, as e_g increases, the ratio e_R/e_g becomes lower, to be almost 1 at $e_g \approx 0.99$. Compared to the previous case (point mass), one can notice a symmetry.

Since the basic intention in introducing e_R and e_z is their use in the description of 3D orbits corre-

sponding to the case of steady state and axial symmetry, there is no reason to insist on coinciding of e_R with e_g in the case of planar motion for a particular type of potential, say that of a point mass, as usually done in the literature. Besides, the point-mass potential can be hardly usable in describing the situation in galaxies. The exceptions may be the very interior, if there is a massive central black hole, and the far periphery. Nevertheless, there is an alternative formulation

$$1 - e_q^2 = \left[1 - \frac{\langle \dot{r}^2 \rangle_t}{\langle v_t^2 \rangle_t} \right]^2. \quad (10)$$

This e_q eccentricity is always equal to e_g for the case of the point-mass potential.

The most important in the present paper is to have a sufficiently simple relation which defines two eccentricities e_R and e_z and, at the same time, makes it possible to introduce the additional eccentricity e_s . The eccentricity e_s may be useful in the local kinematical studies of the Milky Way, because for the thin disc both e_R and e_z are very low and the resulting e_s should also be very low. Reversely, for the halo both e_R and e_z are expected to be in general very high which results in a very high e_s . In the case of a thick disc we have an intermediate situation. Therefore, we may expect to strengthen the kinematical differences among these three entities by including e_s as an additional parameter. Note that in the conditions of the local Milky-Way kinematics $e_s = 0$ automatically determines the modulus of J_z , since all the stars of a sample are practically at the same galactocentric position. The only difference can be in the sign of J_z . However, in the solar neighbourhood stars not orbiting the centre of the Milky Way in the sense of the galactic rotation are extremely rare.

5. CONCLUSION

The orbital eccentricities proposed above (formulae (5) and (6)) may be useful in describing the orbital motion for 3D orbits corresponding to the case of steady state and axial symmetry. They can be easily calculated since the quantities for which the mean over time is obtained are calculated at each step of the numerical procedure. A suggestion for the future work is to use them in the statistical studies concerning the motion of stars in the solar neighbourhood.

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НОВА КИНЕМАТИЧКА ДЕФИНИЦИЈА ЕКСЦЕНТРИЧНОСТИ ПУТАЊЕ

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Оригинални научни рад

Изложено је ново схватање путањске ексцентричности. Бездименционе величине предложене у овом раду да послуже као путањске ексцентричности имају кинематичку природу. Сврха је да се оне користе за описивање кретања за случај тродимензионих ор-

бита. Поређење за скоро раванске путање показује да се вредности ексцентричности предложених овде не разликују знатно од оних које одговарају ексцентричностима геометријске природе које се обично примењују.